BRIEF REPORTS

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Brownian motion in a singular potential and a fractal renewal process

H. F. Ouyang and Z. Q. Huang
*Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China**

E. J. Ding
China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, China;
Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China;
and Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China
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We have proposed a model for the one-dimensional Brownian motion of a single particle in a singular potential field in our previous paper [Phys. Rev. E **50**, 2491 (1994)]. In this Brief Report, we further discuss this model and show that, in some special cases, the Brownian motion can be considered as a finite-valued alternating renewal process, which has been investigated by Lowen and Teich [Phys. Rev. E **47**, 992 (1993)]. The numerical results here are in agreement with those drawn by Lowen and Teich.

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In our previous paper [1], we have proposed a simple one-dimensional model about the Brownian motion of a single particle in a central potential and its generalization. The model is described by the following equation:

$$\dot{x} = -\frac{c\,\mathrm{sgn}(x)}{|\,x\,|^m} + \frac{\Gamma(t)}{x^n} \quad \text{for} \quad x \neq 0, \tag{1}$$

where c, m, and n are constants and $\Gamma(t)$ is a Gaussian white noise with properties

$$\langle \Gamma(t) \rangle = 0$$

and

$$\langle \Gamma(t)\Gamma(t') \rangle = 2D\delta(t-t'),$$

where D is a constant. Without loss of generality, c is taken to be unity. The case of $m = 2$ and $n = 1$ has been discussed in detail in Ref. [1], and in this case the power spectrum of the position x has the form of $1/f^\alpha$ with $\alpha \approx 1$. In this Brief Report, we discuss the case of $m = n+1$, with n being relatively large. The results show that in this case the Brownian motion can be considered as a finite-valued alternating renewal process, which has been investigated by Lowen and Teich [2].

When $m = n + 1$, Eq. (1) assumes the following form:

$$\dot{x} = -\frac{\mathrm{sgn}(x)}{|\,x\,|^{n+1}} + \frac{\Gamma(t)}{x^n} \quad \text{for} \quad x \neq 0. \tag{2}$$

From Eq. (2), we can see that, with n being relatively large, the values and the oscillations of $\dot{x}$ must be very small when x is larger than 1. In other words, in this case, the motion of the particle is relatively quiescent and long-time correlated. The corresponding Fokker-Planck equation for $x > 0$ is

$$\frac{\partial}{\partial t}W(x,t) = -\left[\frac{\partial}{\partial x}D^{(1)}(x) - \frac{\partial^2}{\partial x^2}D^{(2)}(x)\right]W(x,t) \tag{3}$$

with

$$D^{(1)}(x) = -\frac{1}{x^{n+1}} - \frac{Dn}{x^{2n+1}}, \qquad D^{(2)}(x) = \frac{D}{x^{2n}}.$$

Here $W(x,t)$ is the probability density function, and $D^{(1)}(x)$, $D^{(2)}(x)$ are the drift and diffusion coefficients, respectively. Obviously, the above equation has a stationary solution, which has the form

$$W(x) \sim x^n \exp\left(-\frac{x^n}{Dn}\right) \quad \text{for} \quad x > 0. \tag{4a}$$

Similarly, we can also get the solution for $x < 0$

$$W(x) \sim (-x)^n \exp\left(-\frac{(-x)^n}{Dn}\right) \quad \text{for} \quad x < 0. \tag{4b}$$

From Eqs. (4a) and (4b) we can see that the probability

*Mailing address.

$W(x)$ has two sharp peaks at $x = \pm 1$ when n is very large. In order to describe the motion of the particle clearly and conveniently, we denote the state of $x \simeq 1$ as state 1 and the state of $x \simeq -1$ as state 2. Because of the existence of the noise term in Eq. (1) and the symmetry of the potential, the particle must oscillate between the state 1 and state 2. So, the motion of the particle may be approximately regarded as a finite-valued alternating renewal process in this case. The following numerical calculation also shows that the process of motion is a fractal renewal process and the power spectrum of the position x has the form of $1/f^{\alpha}$ with $\alpha \approx 1.5$.

Using the numerical algorithm in Ref. [1], we can calculate from Eq. (2) the position x as a function of time t. By squaring the fast Fourier transformation of the position $x(t)$ we can obtain the power spectrum $S(f)$ of the position $x(t)$. In the numerical calculation, after a long-time transient, the fluctuations of the position $x(t)$ and its corresponding power spectrum $S(f)$ are independent of the initial position $x(0)$ of the particle.

Now let us examine the fluctuations of position $x(t)$. In the numerical calculation, we choose $D = 0.1$ and record the positions $x(t)$ every time interval of 0.0002. (The relationships among D, the fluctuations, and the power spectrum have been discussed in Ref. [1], so we do not repeat them here.) Figure 1 shows the fluctuations of $x(t)$ for the case $m = n + 1 = 64$. This figure only contains 1024 data points of the time sequence of $x(t)$. From Fig. 1 we can see that the particle is either in state 1 or state 2 and alternates between these two states now and then. Meanwhile, only when $|x|$ is slightly larger than 1, can the dwelling time during which the particle stays in state 1 or 2 be long. When the particle is far away from state 1 or 2, the particle will rapidly go back to one of those two states. In addition, we can also see that there exist dwelling times of all scales, with the resolution limited only by the calculating time. To show this, the density $D(t)$ of the dwelling time t during which the particle stays in one state before alternating to another one is calculated. In Fig. 2, we present the plot of $D(t)$ versus

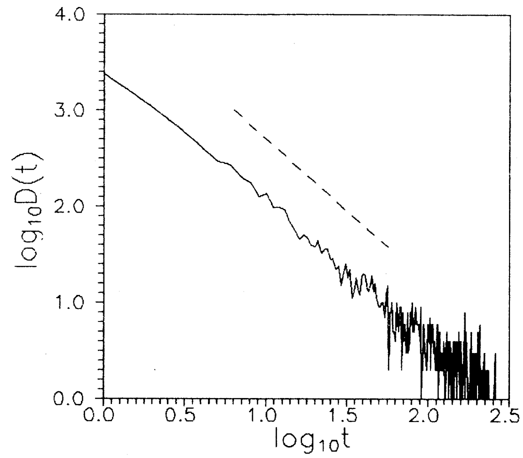

FIG. 2. The density $D(t)$ of the dwelling time t decays in a power-law form; here we have taken the time interval of 0.0002 as the unit of the time. The dashed line has a slope of -1.5.

t; here t is the dwelling time, and we have taken 0.0002 as the unit of the time. From this figure we can see that there exists a power-law relation

$$D(t) \propto t^{-\beta}; \tag{5}$$

here β is about 1.5. In Fig. 3 the relations between $D(t)$ and t for the cases of $m = n + 1 = 8,\ 16,\ 32,\ 64,\ 128$ are given. One can see that the slopes of different curves are almost equal to each other. The corresponding power spectrum is shown in Fig. 4. In this plot, each curve is obtained by averaging 10 samples, and smoothed by averaging over 0.05 unit of $\ln f$. In order to exhibit the power spectra of the position x over a wide range of frequency, every time sequence of $x(t)$ contains a total of 262 144 data points. From Fig. 4 we can see that the power spectra for different cases of n all behave as $1/f^{\alpha}$ with α being almost the same for different cases. In addition, the sum of α and β satisfies $\alpha + \beta \simeq 3.0$, which

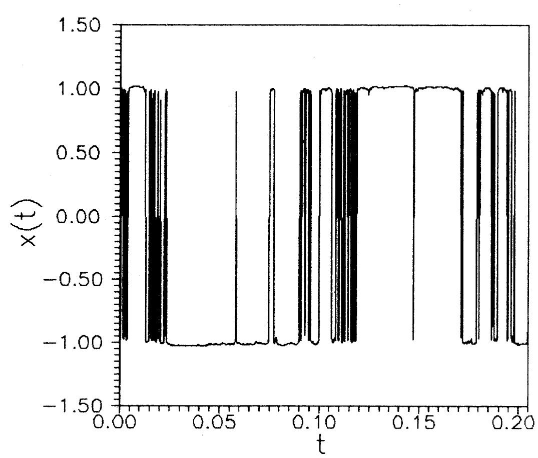

FIG. 1. The fluctuations of the position $x(t)$ when $D = 0.1$ and $m = n+1 = 64$; here both $x(t)$ and t are dimensionless and only the first 1024 data points are shown.

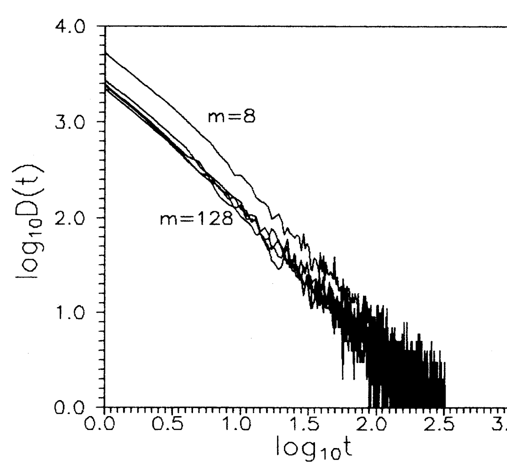

FIG. 3. The densities $D(t)$ of the dwelling time t for the cases of $m = n + 1 = 8,\ 16,\ 32,\ 64,\ 128$. The time unit here is the same as that in Fig. 2.

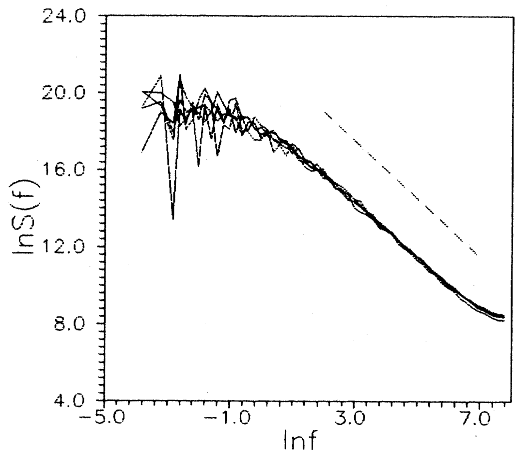

FIG. 4. The power spectra of the position for the cases of $m = n + 1 = 8$, 16, 32, 64, 128. The dashed curve has a slope of -1.5.

is in agreement with the theoretical result of Ref. [2]. With the power-law form for the density $D(t)$, the authors of Ref. [2] demonstrated that some other quantities also followed power-law forms, which they believed to be an indication of the fractal nature of the motion of the particle.

As we have shown in Ref. [1], when $m = 2$, the model can be regarded as a description of motion of the particle in a central potential. In this case the motion of the particle is relatively complex and the power spectrum of the position x has the form of $1/f$. When m becomes large, the motion of the particle becomes very simple and is a fractal renewal process. Based on the results of Ref. [2], this model might propose an explanation of the $1/f$ noise in amorphous semiconductors, etc.

In conclusion, we have considered in this Brief Report a special case of the simple model for a one-dimensional Brownian motion of the single particle in a singular potential proposed in Ref. [1]. Numerical calculations show that the process of the Brownian motion of the particle is a fractal renewal process, and that the relation between the exponents α and β is in agreement with the theoretical one drawn in Ref. [2]. So, we show how a singular potential and a white-noise driving function combine to produce fractal behavior.

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[1] H. F. Ouyang, Z. Q. Huang, and E. J. Ding, Phys. Rev. E **50**, 2491 (1994), and references therein.

[2] S. B. Lowen and M. C. Teich, Phys. Rev. E **47**, 992 (1993).